Horizons and Geodesics of Black Ellipsoids with Anholonomic Conformal Symmetries

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Abstract

The horizon and geodesic structure of static configurations generated by anisotropic conformal transforms of the Schwarzschild metric is analyzed. We construct the maximal analytic extension of such off–diagonal vacuum metrics and conclude that for small deformations there are different classes of vacuum solutions of the Einstein equations describing "black ellipsoid" objects [1]. This is possible because, in general, for off–diagonal metrics with deformed non–spherical symmetries and associated anholonomic frames the conditions of the uniqueness black hole theorems do not hold.

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1 Introduction:

A present research direction is connected to elaboration of new methods of constructing exact solutions of the Einstein equations parametrized by off–diagonal metric ansatz and generated by anholonomic transforms and associated nonlinear connection structures considered on (pseudo) Riemannian space–times [1, 2]. Such vacuum solutions may be constructed in the framework of the Einstein gravity theory of arbitrary dimension. They posses a generic local anisotropy and may describe, for instance, static

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black hole and cosmological solutions with ellipsoidal or toroidal symmetry, various soliton–dilaton configurations and wormholes and flux tubes with anisotropic polarizations and/or running constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy. The investigation of physical properties of new classes of solutions presents a substantial interest.

The aim of this paper is to prove that in the framework of general relativity theory there are static solutions with deformed spherical symmetries (e. g. to the symmetry of resolution ellipsoid) which may describe different types of black hole like objects with non–spherical horizons. In Ref. [2] we analyzed the horizon and geodesic structure of a class of static black ellipsoid solutions and concluded that for some parametrizations and small eccentricity they describe classes of Schwarschild like vacuum metrics with "slightly" deformed horizons. Far a way from the horizons, such generic off–diagonal metrics with ellipsoid symmetry may transform into some diagonal ones with spherical symmetry which results in a compatibility with the asymptotic Minkowski space-time.

For the vacuum exact solutions with spherical symmetry, there were proved the uniqueness black hole theorems (UBHT) [3] which define the basic properties of static and/or stationary gravitational configurations. Such objects are described by metrics which are diagonal or may be diagonalized with respect to a local coordinate coordinate frame, $e_\alpha = \partial_\alpha = \partial/\partial u^\alpha$, satisfying the holonomy relations

$$\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0,$$

where $u^\alpha = (x^i, y^a)$ are local coordinates on a four dimensional pseudo–Riemannian space–time V^{3+1}. The new classes of static and stationary solutions of the vacuum Einstein equations with ellipsoid and/or toroidal symmetry [1, 4] are parametrized by off–diagonal metric ansatz which can be diagonalized only with respect to anholonomic frames

$$e_\alpha = A_\alpha^\beta \left(u^\gamma\right) \partial_\beta, \qquad (1)$$

subjected to anholonomy relations

$$e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma \left(u^\varepsilon\right) e_\gamma, \qquad (2)$$

where $W_{\alpha\beta}^\gamma \left(u^\varepsilon\right)$ are called the coefficients of anholonomy. We illustrated that a very large class of off–diagonal metrics with coefficients depending on two and three variables may be diagonalized by corresponding anholonomic transforms (1) and proved that the vacuum Einstein equations (as well with matter fields for a very general type of energy–momentum tensors) with anholonomic variables admit exact solutions. Such metrics posses a generic anisotropy being constructed in the framework of general gravity or extra/lower dimension Einstein gravity theories. The new type of exact solutions depends anisotropically on a coordinate $y^3 = v$ (for static or stationary configurations chosen as an angular coordinate) and can be generated by some inter–related anholonomic and conforma transforms of usual diagonal locally isotropic solutions (for instance, by some deformations of the Scwarzschild or Reissner–Nordstrom solutions).

We emphasize that for generic off–diagonal metrics and anholonomic gravitational systems the conditions and proofs of UBHT (if such ones can be formulated for some particular anisotropic configurations) have to be revised. We base our arguments on

consequences of the Kaluza–Klein gravity: It is known that for some particular off–diagonal metric ansatz and corresponding re–definitions of Lagrangians we can model a class of effective gravitational (with diagonal metrics) and matter fields (electromagnetic ones or, for higher than five dimensions, non–Abelian gauge fields) interactions. Similar conclusions hold true if there are considered three dimensional gravitational and matter field interactions modeled effectively in the framework of four dimensional gravity. It is obvious that in this case we are not restricted by the conditions of UBHT because we have additional matter field interactions induced from extra dimensions or by off–diagonal terms of the metric in the same dimension. Of course, in general, the vacuum gravitational dynamics of off–diagonal metric coefficients can not be associated to any matter field contributions; the may be of geometric, vacuum gravitational nature, when some degrees of gravitational dynamics are anholonomic (constrained). But, even for such configurations, we have to revise the conditions of application of UBHT if the off–diagonal metrics, anholonomic frames and non–spherical symmetries are considered. The extended non–diagonal dynamics of vacuum gravitational fields subjected to some anholonomy conditions is very different from that analyzed in the cases of the well known Scwarzschild, Kerr–Newmann and Reissner–Nordstrom black hole solutions.

The geometry of pseudo-Riemannian space–times with off–diagonal metrics may be equivalently modeled by some diagonalized metrics with respect to anholnomic frames with associated nonlinear connection structure (see details in Refs. [1, 4]; for nonlinear connections in vector and spinor bundles and in superbundles the formalism was developed in Ref. [5]). In this case we obtain an effective torsion (a "pure" frame effect vanishing with respect to holonomic frames) which together with the anholonomy coefficients modify the formulas for the Levi Civita connection and the corresponding Riemann, Ricci and Einstein tensors by elongating them with additional terms. As a result, for such configurations we can not prove the UBHT. That why it was possible to construct in Refs. [1, 4] various classes of off–diagonal exact vacuum solutions with the same symmetry of the metric coefficients (for instance, ellipsoid or toroidal configurations) and to suggest that some of them can describe certain static black hole objects with non–spherical symmetries which was confirmed by an analysis of geodesic congruences and of horizons in Ref. [2].

In this work, we show that there are alternative possibilities to construct "deformed" black holes by considering conformal transforms adapted to the nonlinear connection structure. We analyze the maximal analytic extension and the Penrose diagrams of such metrics and state the basic properties by comparing them with the Reissner–Nordstrom solution.

The paper has the following structure: in section 2 we present the necessary formulas on off–diagonal metrics with conformal factors which can be diagonalized with respect to anholonomic frames with associated nonlinear connection structure and write down the vacuum Einstein equations. In section 3 we define a class of static anholonomic conformal transforms of the Scwarzschild metric to some off–diagonal metrics. In section 4 we construct the maximal analytic extension of such metrics, for static small ellipsoid deformations, analyze their horizon structure. Section 5 contains a study of the conditions when the geodesic behaviour of ellipsoidal metrics can be congruent to

the conformal transform of the Schwarzschild one. A discussion and conclusions are considered in section 6.

2 Off–diagonal Metrics and Anholonomic Conformal Transforms

Let us consider the quadratic line element

$$ds^2 = \Omega\left(x^i, v\right)\hat{g}_{\alpha\beta}\left(x^i, v\right) du^\alpha du^\beta, \quad (3)$$

with $\hat{g}_{\alpha\beta}$ parametrized by the ansatz

$$\begin{bmatrix} g_1 + {w_1}^2 h_3 + {n_1}^2 h_4 & w_1 w_2 h_3 + n_1 n_2 h_4 & w_1 h_3 & n_1 h_4 \\ w_1 w_2 h_3 + n_1 n_2 h_4 & g_2 + {w_2}^2 h_3 + {n_2}^2 h_4 & w_2 h_3 & n_2 h_4 \\ w_1 h_3 & w_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix}, \quad (4)$$

for a 4D pseudo–Riemannian space–time V^{3+1} enabled with local coordinates $u^\alpha = (x^i, y^a)$ where the indices of type $i, j, k, ...$ run values 1 and 2 and the indices $a, b, c, ...$ take values 3 and 4; $y^3 = v = \varphi$ and $y^4 = t$ are considered respectively as the "anisotropic" and time like coordinates. The components $g_i = g_i\left(x^i\right), h_a = h_{ai}\left(x^k, v\right)$ and $n_i = n_i\left(x^k, v\right)$ are some functions of necessary smoothly class or even singular in some points and finite regions. So, the g_i–components of our ansatz depend only on "holonomic" variables x^i and the rest of coefficients may also depend on one "anisotropic" (anholonomic) variable $y^3 = v$; the ansatz does not depend on the time variable $y^4 = t :$ we shall search for static solutions.

The element (3) can be diagonalized,

$$ds^2 = \Omega\left(x^i, v\right)\left[g_1\left(dx^1\right)^2 + g_2\left(dx^2\right)^2 + h_3\left(\delta v\right)^2 + h_4\left(\delta y^4\right)^2\right], \quad (5)$$

with respect to the anholomic co–frame

$$\delta^\alpha = (d^i = dx^i, \delta^a = dy^a + N_i^a dx^i) = \left(d^i, \delta v = dv + w_i dx^i, \delta y^4 = dy^4 + n_i dx^i\right) \quad (6)$$

which is dual to the frame

$$\delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_b) = (\delta_i = \partial_i - w_i\partial_3 - n_i\partial_4, \partial_3, \partial_4), \quad (7)$$

where $\partial_i = \partial/\partial x^i$ and $\partial_b = \partial/\partial y^b$ are usual partial derivatives. The tetrads δ_α and δ^α are anholonomic because, in general, they satisfy some anholonomy relations (2) with non–trivial coefficients

$$W_{ij}^a = \delta_i N_j^a - \delta_j N_i^a,\ W_{ia}^b = -\ W_{ai}^b = \partial_a N_i^b.$$

The anholonomy is induced by the coefficients $N_i^3 = w_i$ and $N_i^4 = n_i$ which "elongate" partial derivatives and differentials if we are working with respect to anholonomic

frames. They define a nonlinear connection (in brief, N–connection) associtated to an anholonomic frame structure with mixed holonomic (x^i) and anholonomic (y^a) coordinates, see details in Refs. [1, 4, 5]. Here we emphasize that on (pseudo) Riemannian spaces the N–connection is an object completely defined by anholonomic frames, when the coefficients of tetradic transform (1), $A_\alpha^\beta(u^\gamma)$, are parametrized explicitly via values $\left(N_i^a, \delta_i^j, \delta_b^a\right)$, where δ_i^j and δ_b^a are the Kronecker symbols.

By straightforward calculation with respect to the frames (6)–(7) [4, 1, 2] we find the non–trivial components of the Ricci tensor and of the vacuum Einstein equations,

$$R_1^1 = R_2^2 = -\frac{1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'}g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}] = 0, \quad (8)$$

$$R_3^3 = R_4^4 = -\frac{\beta}{2h_3h_4} = -\frac{1}{2h_3h_4}\left[h_4^{**} - h_4^{*}\left(\ln\sqrt{|h_3h_4|}\right)^{*}\right] = 0, \quad (9)$$

$$R_{3i} = -w_i\frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0, \quad (10)$$

$$R_{4i} = -\frac{h_4}{2h_3}\left[n_i^{**} + \gamma n_i^{*}\right] = 0, \quad (11)$$

for

$$\delta_i h_3 = 0 \quad (12)$$

and

$$\delta_i \Omega = 0 \quad (13)$$

where

$$\alpha_i = \partial_i h_4^{*} - h_4^{*}\partial_i \ln\sqrt{|h_3h_4|},\ \gamma = 3h_4^{*}/2h_4 - h_3^{*}/h_3, \quad (14)$$

and the partial derivatives are denoted $g_1^{\bullet} = \partial g_1/\partial x^1, g_1^{'} = \partial g_1/\partial x^2$ and $h_3^{*} = \partial h_3/\partial v$. We imposed additionally the condition $\delta_i N_j^a = \delta_j N_i^a$ in order to work with the Levi Civita connection (see a discussion in [2]) which may be satisfied, for instance, if

$$w_1 = w_1\left(x^1, v\right), n_1 = n_1\left(x^1, v\right), w_2 = n_2 = 0, \quad (15)$$

or, inversely, if

$$w_1 = n_1 = 0, w_2 = w_2\left(x^2, v\right), n_2 = n_2\left(x^2, v\right).$$

We note that the condition (13) relates the conformal factor Ω with the coefficients of the N–connection and with another off–diagonal metric components, which are also subjected to the conditions (12). This induces a constrained type, aholonomic (anisotropic), of conformal symmetry.

The system of equations (8)–(11) and the conditions (12)–(13) can be integrated in general form [4]. Physical solutions are defined following some additional boundary conditions, imposed types of symmetries, nonlinearities and singular behaviour and compatibility in the locally anisotropic limits with some well known exact solutions.

3 Anholonomic Conformal Deformations of the Schwarzschild Metric

We consider a particular parametrization of metrics with anisotropic conformal factors and non–spherical (in particular, ellipsoidal) symmetry defined in Refs [4, 1],

$$\begin{aligned} \delta s^2 &= \Omega(r,\varphi)[-\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right)^{-1}dr^2 - r^2 q(r)d\theta^2 \\ &\quad -\eta_3(r,\theta,\varphi)r^2\sin^2\theta\delta\varphi^2 + \left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right)\delta t^2] \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Omega &= 1+\varepsilon\lambda(r,\varphi)+\varepsilon^2\gamma(r,\varphi), \\ \delta\varphi &= d\varphi + w_1(r,\varphi)dr \text{ and } \delta t = dt + n_1(r,\varphi)dr, \end{aligned}$$

for w_1 and $n_1 \sim \varepsilon$. In this paper we shall not work with exact solutions but, for simplicity, we shall consider some conformal and anholonomic deformations of the Schwarzschild metric up to the second order on ε, considered as a small parameter $(0 \leq \varepsilon \ll 1)$, which are contained in those exact solutions. The functions $q(r), \lambda(r,\varphi)$, $\gamma(r,\varphi)$, $\eta_3(r,\varphi)$, $w_1(r,\varphi)$ and $n_1(r,\varphi)$ will be found as the metric (16) would define a solution of the vacuum Einstein equations (in the limit $\varepsilon \to 0$ and $\eta_3 \to 1$ we have just the Schwarzschild solution for a point particle of mass m).

We note that the condition of vanishing of the metric coefficient before δt^2,

$$\left(1+\varepsilon\lambda+\varepsilon^2\gamma\right)\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right) = 1-\frac{2m}{r}+\varepsilon\frac{\Phi}{r^2}+\varepsilon^2\Theta = 0, \tag{17}$$

where

$$\begin{aligned} \Phi &= \lambda\left(r^2-2mr\right)+1, \\ \Theta &= \gamma\left(1-\frac{2m}{r}\right)+\lambda \end{aligned}$$

defines, for instance, a rotation ellipsoid configuration if λ and γ are chosen

$$\begin{aligned} \lambda &= \left(1-\frac{2m}{r}\right)^{-1}\left(\cos\varphi-\frac{1}{r}\right), \\ \gamma &= -\lambda\left(1-\frac{2m}{r}\right)^{-1}. \end{aligned} \tag{18}$$

We note that all computations should be performed up to the second order on ε, but because we may fix some conditions on λ and γ as to make zero the coefficient defining deformations of horizons which are proportional to ε^2, we may restrict our further analysis only to linear on ε deformations.

A zero value of the coefficient (17), with ellipsoid like symmetry stated by the values (18), in the first order on ε, follows if

$$r_+ = \frac{2m}{1+\varepsilon\cos\varphi} = 2m[1-\varepsilon\cos\varphi], \tag{19}$$

which is the equation for a 3D ellipsoid like hypersurface with a small eccentricity ε. In general, we can consider any arbitrary functions $\lambda(r,\theta,\varphi)$ and $\eta_3(r,\theta,\varphi)$.

Having introduced a new radial coordinate

$$\xi = \int dr\sqrt{|1-\frac{2m}{r}+\frac{\varepsilon}{r^2}|} \tag{20}$$

and defined

$$\begin{aligned} h_3 &= -\eta_3(\xi,\theta,\varphi)r^2(\xi)\sin^2\theta, \\ h_4 &= 1-\frac{2m}{r}+\frac{\varepsilon}{r^2}, \end{aligned} \tag{21}$$

for $r=r\left(\xi\right)$ found as the inverse function after integration in (20), we write the metric (16) in coordinates $\left(\xi,\theta,\varphi,t\right)$,

$$\begin{aligned} ds^2 &= \Omega\left(\xi,\theta,\varphi\right)\left[-d\xi^2 - r^2\left(\xi\right)q(\xi)d\theta^2 + h_3\left(\xi,\theta,\varphi\right)\delta\varphi^2 + h_4\left(\xi\right)\delta t^2\right], \\ \delta\varphi &= d\varphi + w_1\left(\xi,\varphi\right)d\xi \text{ and } \delta t = dt + n_1\left(\xi,\varphi\right)d\xi, \quad r^2\left(\xi\right)q(\xi) = \xi^2, \end{aligned} \tag{22}$$

where the coefficient n_1 is taken to satisfy (15).

Let us find the conditions when the coefficients of the metric (16) define solutions of the vacuum Einstein equations. For $g_1=-1, g_2=-r^2\left(\xi\right)q(\xi)$ and arbitrary $h_3(\xi,\theta,\varphi)$ and $h_4\left(\xi\right)$ solve the equations (8)–(10). The coefficients n_i are solutions of the equations (11) with $h_4^*=0$,

$$n_k = n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int \eta_3(\xi,\theta,\varphi)d\varphi,\ \ h_3^*\neq 0; \tag{23}$$

the functions $n_{k[1,2]}\left(x^i\right)$ are to be stated by boundary conditions. In order to consider linear infinitesimal extensions on ε of the Schwarzschild metric we write

$$n_1 = \varepsilon\widehat{n}_1\left(\xi,\varphi\right)$$

where

$$\widehat{n}_1\left(\xi,\varphi\right) = n_{1[1]}\left(\xi\right) + n_{1[2]}\left(\xi\right)\int \eta_3\left(\xi,\varphi\right)d\varphi. \tag{24}$$

The last step is to find such $\Omega\left(\xi,\varphi\right)$, w_i and $\eta_3\left(\xi,\theta,\varphi\right)$ as to satisfy (12)–(13). In the first order on ε those equations may be solved by

$$\Omega = 1+\varepsilon h_3$$

which imposes the condition

$$-\eta_3\left(\xi,\theta,\varphi\right)r^2\sin^2\theta = \lambda\left(\xi,\varphi\right) + \varepsilon\gamma\left(\xi,\varphi\right),$$

and by

$$
\begin{aligned}
w_i &= \partial_i \Omega/\Omega^*, \eta_3^* \neq 0, \\
&= 0, \eta_3^* = 0.
\end{aligned}
$$

The data

$$
\begin{aligned}
g_1 &= -1, g_2 = -r^2(\xi), \Omega = 1 + \varepsilon\lambda\,(r,\varphi) + \varepsilon^2\gamma\,(r,\varphi)\,, \\
h_3 &= -\eta_3(\xi,\theta,\varphi) r^2(\xi) \sin^2\theta, h_4 = 1 - \frac{2m}{r(\xi)} + \frac{\varepsilon}{r^2(\xi)}, \\
w_1 &= \partial_1 \Omega/\Omega^*, n_1 = \varepsilon[n_{1[1]}\,(\xi) + n_{1[2]}\,(\xi) \int \eta_3\,(\xi,\theta,\varphi)\,d\varphi], \\
w_2 &= 0, n_2 = 0,
\end{aligned} \tag{25}
$$

for the metric (16) define a class of solutions of the vacuum Einstein equations depending on arbitrary functions $\eta_3(\xi,\varphi), n_{1[1]}\,(\xi)$ and $n_{1[2]}\,(\xi)$ which should be defined by some boundary conditions. Such solutions are generated by small deformations (in particular cases of rotation ellipsoid symmetry) of the Schwarzschild metric.

In order to connect our solutions with some small deformations of the Schwarzschild metric, as well to satisfy the asymptotically flat condition, we must chose such functions $n_{k[1,2]}\,(x^i)$ as $n_k \rightarrow 0$ for $\varepsilon \rightarrow 0$ and $\eta_3 \rightarrow 1$. These functions have to be selected as to vanish far away from the horizon, for instance, like $\sim 1/r^{1+\tau}, \tau > 0$, for long distances $r \rightarrow \infty$.

4 Analytic Extensions of Conformal Ellipsoid Metrics

We emphasize that the metric (16) (equivalently (22)) considered with respect to the anholonomic basis has a number of similarities with the Schwrzschild and Reissner–Nordstrem solutions. We can identify ε with e^2 and treat it as a stationary metric with effective "electric" charge induced by small off–diagonal metric extensions. We analyzed such metrics in Ref. [2] where the conformal factor was considered for a trivial value $\Omega = 1$ (the off–diagonal extension being driven by a different equation). The coefficients of such metrics are similar to those from the Reissner–Nordstrem solution but given with respect to a corresponding anholonomic frame and with additional dependencies on "polarization functions" $\eta_3, n_{1[1,2]}\,(\xi)\,, \lambda$ and γ. Nevertheless, there is a substantial difference: the static deformed metric generated by anholnomic transforms satisfies a solution of the vacuum Einstein equations which differs substantially from the Reissner–Nordstrem metric being an exact static solution of the Einstein–Maxwell equations.

For $\varepsilon \rightarrow 0$ and $\eta_3 \rightarrow 1$ the metric (16), as well those considered in Ref. [2], transforms into the usual Schwarzschild metric. We can consider a special type of deformations which induces ellipsoid symmetries. Such symmetries are determined by corresponding conditions of vanishing of the coefficient before δt which describe

ellipsoidal hypersurfaces like for the rotating Kerr metric, but in our case for a static off–diagonal metric.

In the first order on ε the metric (16) has singularities at

$$r_{\pm} = m\left[1 \pm \left(1 - \varepsilon\frac{\Phi(\varphi)}{2m^2}\right)\right],$$

for $\Phi(\varphi)$ stated for $r = 2m$. For simplicity, we may take explicit equations for the hypersurfaces $r_+ = r_+(\varphi)$ and analyze the horizon properties for some fixed "directions" given in a smooth vecinity of any value $\varphi = \varphi_0$ and $r_+^0 = r_+(\varphi_0)$. The metrics (16) and (22) are regular in the regions I ($\infty > r > r_+^0$), II ($r_+^0 > r > r_-^0$) and III ($r_-^0 > r > 0$). As in the Schwarzschild, Reissner–Nordstrem and Kerr cases these singularities can be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [9, 10]. We have similar regions as in the Reissner–Nordstrem space–time, but with just only one possibility $\varepsilon < 1$ instead of three relations for static electro–vacuum cases ($e^2 < m^2, e^2 = m^2, e^2 > m^2$; where e and m are correspondingly the electric charge and mass of the point particle in the Reissner–Nordstrem metric). Saw we may consider the usual Penrose's diagrams as for a particular case of the Reissner–Nordstrem space–time but keeping in mind that such diagrams and horizons may be drawn for a fixed value of the anisotropic angular coordinate.

We may construct the maximally extended manifold in a similar manner as for the anisotropic metrics with trivial conformal factors [2] and to proceed in steps analogous to those in the Schwarzschild case (see details, for instance, in Ref. [7])). At the first step we introduce a new coordinate

$$r^{\|} = \int dr \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1}$$

for $r > r_+^1$ and find explicitly the coordinate

$$r^{\|} = r + \frac{(r_+^1)^2}{r_+^1 - r_-^1}\ln(r - r_+^1) - \frac{(r_-^1)^2}{r_+^1 - r_-^1}\ln(r - r_-^1), \tag{26}$$

where $r_\pm^1 = r_\pm^0$ with $\Phi = 1$. If the coordinate r is expressed as a function on variable ξ, than the coordinate $r^{\|}$ can be also expressed as a function on ξ depending additionally on some parameters.

At the second step one defines the advanced and retarded coordinates, $v = t + r^{\|}$ and $w = t - r^{\|}$, with corresponding elongated differentials

$$\delta v = \delta t + dr^{\|} \text{ and } \delta w = \delta t - dr^{\|}$$

which transform the metric (22) as

$$\delta s^2 = \Omega(\xi,\theta,\varphi)\left[-r^2(\xi)q(\xi)d\theta^2 - \eta_3(\xi,\theta,\varphi)r^2(\xi)\sin^2\theta\delta\varphi^2 + (1 - \frac{2m}{r(\xi)} + \frac{\varepsilon}{r^2(\xi)})\delta v\delta w\right],$$

where (in general, in non–explicit form) $r(\xi)$ is a function of type $r(\xi) = r\left(r^{\|}\right) = r\left(v,w\right).$ The final steps consist in introducing some new coordinates (v'', w'') by

$$v'' = \arctan\left[\exp\left(\frac{r^1_+ - r^1_-}{4(r^1_+)^2}v\right)\right], w'' = \arctan\left[-\exp\left(\frac{-r^1_+ + r^1_-}{4(r^1_+)^2}w\right)\right]$$

and multiplying the last term on the conformal factor Ω. We obtain

$$\begin{aligned} \delta s^2 &= \Omega\left(\xi,\theta,\varphi\right)\left[-r^2 q d\theta^2 - \eta_3(r,\theta,\varphi) r^2 \sin^2\theta \delta\varphi^2\right] \\ &\quad +64\frac{(r^1_+)^4}{(r^1_+ - r^1_-)^2}\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\Phi\left(\varphi\right)\right)\delta v''\delta w'', \end{aligned} \tag{27}$$

where r is defined implicitly by

$$\tan v'' \tan w'' = -\exp\left[\frac{r^1_+ - r^1_-}{2(r^1_+)^2}r\right]\sqrt{\frac{r-r^1_+}{(r-r^1_-)^{\chi}}}, \chi = \left(\frac{r^1_+}{r^1_-}\right)^2.$$

As particular cases, we may consider such $\eta_3\left(r,\theta,\varphi\right)$ as the condition of vanishing of the metric coefficient before $\delta v''\delta w''$ will describe a horizon parametrized by a resolution ellipsoid hypersurface.

The maximal extension of the Schwarzschild metric polarized by a conformal factor Ω and a function η_3 and deformed by a small parameter ε (for ellipsoid configurations treated as the eccentricity), i. e. of the metric (16), is defined by taking (27) as the metric on the maximal manifold on which this metric is of smoothly class C^2. The Penrose diagram of this static but locally anisotropic space–time, for any fixed angular coordinates φ_0 is similar to the Reissner–Nordstrom solution, for the case $e^2 = \varepsilon$ and $e^2 < m^2$ (see, for instance, Ref. [7])). This way we define a locally anisotropic conformal alternative to the locally anisotropic space–time constructed in Ref. [2]. There is an infinite number of asymptotically flat regions of type I, connected by intermediate regions II and III, where there is still an irremovable singularity at $r = 0$ for every region III. It is "possible" to travel from a region I to another ones by passing through the 'wormholes' made by anisotropic deformations (ellipsoid off–diagonality of metrics, or anholonomy) like in the Reissner–Nordstrom universe because $\sqrt{\varepsilon}$ may model an effective electric charge. We can not turn back in a such anisotropic travel.

Finally, in this section, we note that the metric (27) is analytic every were except at $r = r^1_-$. We may eliminate this coordinate degeneration by introducing another new coordinates

$$v''' = \arctan\left[\exp\left(\frac{r^1_+ - r^1_-}{2n_0(r^1_+)^2}v\right)\right], w''' = \arctan\left[-\exp\left(\frac{-r^1_+ + r^1_-}{2n_0(r^1_+)^2}w\right)\right],$$

where the integer $n_0 \geq (r^1_+)^2/(r^1_-)^2$. In these coordinates, the metric is analytic every were except at $r = r^1_+$ where it is degenerate. This way the space–time manifold can be covered by an analytic atlas by using coordinate carts defined by $(v'', w'', \theta, \varphi)$ and $(v''', w''', \theta, \varphi)$. For maximally analytic extensions of anisotropic metrics we must perform also extensions to non–singular values of polarization coefficients.

5 Conformally Anisotropic Geodesics

The analysis of small deformations of geodesics, linear on ε, of the metric (22) with the data (25) is similar to to that presented in Ref. [2] for the anisotropic metrics with trivial conformal factor. We introduce the effective Lagrangian (see, for instance, Ref. [11])

$$
\begin{aligned}
2L &= g_{\alpha\beta}\frac{\delta u^{\alpha}}{ds}\frac{\delta u^{\beta}}{ds} = \Omega\left(r,\varphi\right)\times\{-\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^{2}}\right)^{-1}\left(\frac{dr}{ds}\right)^{2} \\
&\quad -r^{2}q(r)\left(\frac{d\theta}{ds}\right)^{2}-\eta_{3}(r,\theta,\varphi)r^{2}\sin^{2}\theta\left(\frac{d\varphi}{ds}+\frac{\varepsilon}{\eta_{3}^{*}}\eta_{3}^{\bullet}\frac{dr}{ds}\right)^{2} \\
&\quad +\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^{2}}\right)[\frac{dt}{ds}+\varepsilon\left(n_{1[1]}(r)+n_{1[2]}(r)\int\eta_{3}(r,\varphi)d\varphi\right)\frac{dr}{ds}]^{2}\},
\end{aligned}
\tag{28}
$$

for $\eta_3^{\bullet}=\partial\eta_3/\partial r$ and $r=r(\xi)$.

The corresponding Euler–Lagrange equations,

$$
\frac{d}{ds}\frac{\partial L}{\partial\frac{\delta u^{\alpha}}{ds}}-\frac{\partial L}{\partial u^{\alpha}}=0
$$

are

$$
\begin{aligned}
&\frac{d}{ds}\left[-r^{2}q(r)\Omega\frac{d\theta}{ds}\right] = -\eta_{3}r^{2}\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^{2}+ \\
&\varepsilon\eta_{3}r^{2}\sin\theta\cos\theta\left[(1-\frac{2m}{r}+\frac{\varepsilon}{r^{2}})\left(\frac{dt}{ds}\right)^{2}-\left(\frac{d\xi}{ds}\right)^{2}-2\frac{\eta_{3}^{\bullet}}{\eta_{3}^{*}}\frac{d\xi}{ds}\frac{d\varphi}{ds}-r^{2}q\left(\frac{d\theta}{ds}\right)^{2}\right],
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\frac{d}{ds}\left[-\eta_{3}r^{2}(\frac{d\varphi}{ds}+\varepsilon\frac{\eta_{3}^{\bullet}}{\eta_{3}^{*}}\frac{d\xi}{ds})\Omega\right] = -\eta_{3}^{*}\frac{r^{2}}{2}\sin^{2}\theta\left(\frac{d\varphi}{ds}\right)^{2}+ \\
&\varepsilon\{\left(1-\frac{2m}{r}\right)n_{1[2]}(\xi)\eta_{3}\frac{d\xi}{ds}\frac{d\theta}{ds}+\frac{\eta_{3}^{*}}{2}r^{2}\sin^{2}\theta[\left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^{2}-\left(\frac{d\xi}{ds}\right)^{2} \\
&-r^{2}q\left(\frac{d\theta}{ds}\right)^{2}-\eta_{3}(1+r^{2}\sin^{2}\theta)\left(\frac{d\varphi}{ds}\right)^{2}-(\frac{\eta_{3}^{\bullet}}{\eta_{3}^{*}}-\frac{\eta_{3}}{\eta_{3}^{*}}\left(\frac{\eta_{3}^{\bullet}}{\eta_{3}^{*}}\right)^{*})\frac{d\xi}{ds}\frac{d\varphi}{ds}]\},
\end{aligned}
\tag{30}
$$

and

$$
\frac{d}{ds}\left\{(1-\frac{2m}{r}+\frac{\varepsilon}{r^{2}})\Omega\left[\frac{dt}{ds}+\varepsilon\left(n_{1[1]}+n_{1[2]}\int\eta_{3}d\varphi\right)\frac{d\xi}{ds}\right]\right\}=0,
\tag{31}
$$

where we have omitted the variations for $d\xi/ds$ which may be found from (28) and considered that the polarization η_3 is taken as to have h_3 depending only on variables r and φ. The system of equations (29)–(31) transform into the usual system of geodesic equations for the Schwarzschild space–time if $\varepsilon\to 0$ and $\eta_3\to 1$ which can be solved exactly [11]. For nontrivial values of the parameter ε, conformal factor Ω and polarization η_3, it is a cumbersome task to construct explicit solutions. We do not solve this problem in the present work. We conclude only that from the equation (31) one

follows the existence of an energy like integral of motion, $E = E_0 + \varepsilon E_1$, with

$$E_0 = \left(1 - \frac{2m}{r}\right) \frac{dt}{ds}$$
$$E_1 = \left(\frac{1}{r^2} + r^2 \eta_3\right) \frac{dt}{ds} + \left(1 - \frac{2m}{r}\right) \left(n_{1[1]} + n_{1[2]} \int \eta_3 d\varphi\right) \frac{d\xi}{ds}.$$

The anisotropic conformal deformations of the congruences of Schwarzschild's space–time geodesics preserve the known behaviour in the vicinity of the horizon hypersurface defined by the condition of vanishing of the coefficient $1 - 2m/r + \varepsilon \Phi(r, \varphi)/r^2$ in (27). We can prove this by considering radial null geodesics in the "equatorial plane" satisfying the condition (28) with $\theta = \pi/2, d\theta/ds = 0, d^2\theta/ds^2 = 0$ and $d\varphi/ds = 0$, when

$$\frac{dr}{dt} = \pm \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right) \left[1 + \varepsilon \left(n_{1[1]} + n_{1[2]} \int \eta_3 d\varphi\right)\right].$$

The integral of this equation is

$$t = \pm r^{\|} + \varepsilon_0 \int \left[n_{1[1]} + n_{1[2]} \int \eta_3 d\varphi\right] dr$$

where the coordinate $r^{\|}$ is defined in equation (26). Even the explicit form of the integral depends on the type of polarizations $\eta_3(r, \theta, \varphi)$ and $n_{1[1,2]}(r)$, which may result in small deviations of the null–geodesics for certain prescribed class of non–singular polarizations, we conclude that for an in–going null–ray the coordinate time t increases from $-\infty$ to $+\infty$ as r decreases from $+\infty$ to r_+^1, decreases from $+\infty$ to $-\infty$ as r further decreases from r_+^1 to r_-^1, and increases again from $-\infty$ to a finite limit as r decreses from r_-^1 to zero. We have a behaviour very similar to that for the Reissner–Nordstrom solution but with some additional anisotropic contributions being proportional to ε. The conformal factor Ω contributes indirectly via modification of the coefficients $n_{1[1,2]}$ in the solutions. We also note that as dt/ds tends to $+\infty$ for $r \to r_+^1 + 0$ and to $-\infty$ as $r_- + 0$, any radiation received from infinity appear to be infinitely red–shifted at the crossing of the event horizon and infinitely blue–shifted at the crossing of the Cauchy horizon.

Following the mentioned properties of null–geodesics, we conclude that the metric (16) (equivalently, (22)) with the data (25) and it maximal analytic extension (27) define an alternative (to that constructed in Ref. [2]) black hole static solution which is obtained by an anisotropic small (on ε) deformation of the Schwarzchild solution (for a particular type of deformations the horizon of such black holes is defined by ellipsoid hypersurfaces). We call such objects as black conformal ellipsoids. They exist in the framework of general relativity as certain vacuum solutions of the Einstein equations defined by static off–diagonal metrics and associated anholonomic frames. This property disinguishes them from similar configurations of Reissner–Norstrom type (which are static electrovacuum solutions of the Einstein–Maxwell equations) and of Kerr type rotating solutions, in the last case also with ellipsoid horizons but parametrized by off–diagonal vacuum metrics constructed for a spherical coordinate system which induces a holonomic local frame.

6 Conclusions

In summary, we proved that there are alternative small static conformal anisotropic deformations of the Schwarschild black hole solution (as particular cases we can consider resolution ellipsoid lconfigurations) which have a horizon and geodesic congruence being slightly deformed from the spherical symmetric case. There are possible different types of parametrization of off–diagonal metrics defining the exact solutions of vacuum Einstein equations which posses ellipsoid symmetries as it was constructed in Refs. [1, 4, 2].

We can generate static ellipsoid black hole configurations with trivial, or non–trivial conformal factors. The corresponding metrics admit maximial analytic extensions and with the Penrose diagrams being similar to the Reissner–Nordstrom solution. There is a similarity with the Reissner–Nordstrom metric if the parameter of a conformal ellipsoid deformation (the eccentricity) is treated as an effective electromagnetic charge induced by off–diagonal vacuum gravitational interactions.

As we concluded in Ref. [2] the static ellipsoid black holes posses spherical topology and satisfy the principle of topological censorship [12]. The anisotropic conformal transforms, emphasized in this paper, do not violate such principles. The existence of alternative classes of black ellipsoid solutions does not contradict the black hole uniqueness theorems [3] which were formulated for spherically symmetric solutions. At asymptotics, at least for a very small eccentricity, the anisotropic metrics transform into the usual Schwarzschild one.

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